

# Simple Moufang loops and alternative algebras

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## Abstract

Let a Moufang loop  $Q$  contain a non-unitary subloop, which is a simple loop. Then  $Q$  is not embedded into a loop of invertible elements of any alternative algebra.

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The alternative algebras satisfy the Moufang identities that define the Moufang loops and the set  $U(A)$  of all invertible elements of any alternative algebra  $A$  forms a Moufang loop with respect to multiplication [1]. Therefore, the question raised in [2] is natural.

**Question.** *Is it true that any Moufang loop can be imbedded into a loop of type  $U(A)$  for a suitable unital alternative algebra  $A$ ?*

In general, the answer to this question is negative. In [3] the Moufang loop  $U(\mathbf{O})/\mathbf{R}^*$  for the algebra  $\mathbf{O}$  of classical Cayley algebra (the Cayley-Dickson algebra  $\mathbf{C}(-1, -1, -1)$  in the notation of [4]) over the real field  $\mathbf{R}$  and the similar loop for the Cayley-Dickson algebra over the finite field  $GF(p^2)$ ,  $p > 2$ , are not imbeddable into the loops of type  $U(A)$ .

The field  $CF(p^2)$  is split. Any quadratic equation with coefficients from  $CF(p)$  is solvable in  $CF(p^2)$ . Then the field  $CF(p^2)$  is closed under root operation and by [5, pag. 475, Theorem] the corresponding Moufang loop for  $CF(p^2)$  is simple.

Now we prove that the Moufang loop  $U(\mathbf{O})/\mathbf{R}^*$  also is simple for classical Cayley algebra  $\mathbf{O}$ . Really, suppose contrary, that the loop  $U(\mathbf{O})/\mathbf{R}^*$  is non-simple. Let  $\overline{H}$  be a proper normal subloop of loop  $U(\mathbf{O})/\mathbf{R}^*$  and let  $H$  be the inverse image of  $\overline{H}$  under homomorphism loops  $U(\mathbf{O}) \rightarrow U(\mathbf{O})/\overline{H}$ . Then  $H$  will be a proper normal subloop of  $U(\mathbf{O})$ .

The Cayley algebra  $\mathbf{O}$  is a 8-dimensional corp [6]. Then  $U(\mathbf{O}) = \mathbf{O} \setminus \{0\}$ . Let  $1 = e_0, e_1, \dots, e_7$  be the canonical basis of  $\mathbf{O}$ . The normal subloop  $H$  of  $U(\mathbf{O})$  is proper. Then  $e_j \notin H$  for some  $e_j \in \{e_0, \dots, e_7\}$ .

Any element in  $\mathbf{O}$  has a form  $\sum_{i=0}^7 \alpha_i e_i$ , where  $\alpha_i \in \mathbf{R}$ . As  $U(\mathbf{O}) = \mathbf{O} \setminus \{0\}$  then the mapping  $\varphi : \sum_{i=0}^7 \alpha_i e_i \rightarrow \sum_{i=0}^7 \alpha_i e_i H$  is a homomorphism of algebra  $\mathbf{O}$ . Obviously,  $H = U(\mathbf{O}) \cap (1 + \ker \varphi)$ . Then from  $e_j \notin H$  it follows that  $e_j - 1 \notin \ker \varphi$ . Hence  $\ker \varphi$  is a proper ideal of  $\mathbf{O}$ , i.e.  $\mathbf{O}$  is a non-simple alternative algebra. But by Kleinfeld Theorem the Cayley-Dickson algebras and only they can be simple alternative algebras [4]. We get a contradiction. Consequently, the Moufang loop  $U(\mathbf{O})/\mathbf{R}^*$  is simple.

The following Theorem together with those proved above generalize the main result from [3]. Let us prove first the following:

**Lemma.** *Let  $I$  be an ideal of alternative algebra  $A$  with unit 1 and let  $H$  be a subloop of Moufang loop  $U(A)$ . Then  $K = H \cap (1 + I)$  is a normal subloop of loop  $H$ .*

**Proof.** The homomorphism of algebras  $A \rightarrow A/I$  induce a homomorphism  $\varphi$  of loop  $H$ . Denote  $\varphi(H, \cdot) = (\overline{H}, \star)$ . Any Moufang loop is a  $IP$ -loop, i.e. satisfies the identities  $x^{-1} \cdot xy = y$ ,  $yx \cdot x^{-1} = y$ , where  $x^{-1}x = xx^{-1} = 1$ . From identity  $x^{-1} \cdot xy = y$  it follows that  $\varphi(x^{-1}) = (\varphi x)^{-1}$  and  $(\varphi x^{-1}) \star (\varphi x \star \varphi y) = \varphi y$ ,  $(\varphi x)^{-1} \star (\varphi x \star \varphi y) = \varphi y$ ,  $\overline{x}^{-1} \star (\overline{x} \star \overline{y}) = \overline{y}$ .

Let  $\overline{a}, \overline{b} \in \overline{H}$ . It is obvious that the equation  $\overline{a} \star x = \overline{b}$  is always solvable and as  $\overline{a}^{-1} \star (\overline{a} \star x) = \overline{a}^{-1} \star \overline{b}$ ,  $x = \overline{a}^{-1} \star \overline{b}$ , then it is uniquely solvable. It can be shown by analogy that the equation  $y \star \overline{a} = \overline{b}$  is also uniquely solvable. Hence  $(\overline{H}, \star)$  is a loop. Then  $\ker \varphi = H \cap (1 + I)$  is a normal subloop of loop  $H$ , as required.

**Theorem.** *Let a Moufang loop  $Q$  contain a non-unitary subloop, which is a simple loop. Then the loop  $Q$  is not imbedded into the loop of type  $\mathcal{U}(A)$  for a suitable unital alternative  $F$ -algebra  $A$ , where  $F$  is an associative commutative ring with unit.*

**Proof.** Obviously, it is sufficient to consider that  $Q$  is a simple Moufang loop. In process of proof will be used without reference some definitions and results from theory of alternative algebras from [4].

Assume that  $Q$  is imbedded into a loop  $\mathcal{U}(A)$  for a certain alternative algebra  $A$ . We will identify the elements from  $Q$  with their images in  $Q$ .

Let  $F\{Q\}$  be the submodule of  $F$ -module  $A$  generated by set  $\{g | g \in Q\}$ . Finite sums  $\sum_{g \in Q} \alpha_g g$ , where  $\alpha_g \in F$ , and only they are elements in  $F\{Q\}$ . Obviously,  $F\{Q\}$  is a subalgebra of algebra  $A$ ,  $Q$  is a subloop of loop  $U(F\{Q\})$ . If  $I$  is a proper ideal of  $F\{Q\}$  then  $g \notin I$  for some  $g \in Q$ . In this case by Lemma the ideal  $I$  of  $F\{Q\}$  induces the proper normal subloop  $K = Q \cap (1 + I)$  of loop  $Q$ .

Let  $I_1, I_2$  be a proper ideals of algebra  $F\{Q\}$  and let  $K_1, K_2$  be the proper normal subloops of loop  $Q$  corresponding to ideals  $I_1, I_2$  by Lemma. The sum  $I_1 + I_2$  is the minimal ideal of  $F\{Q\}$  containing the ideals  $I_1, I_2$ , the product  $K_1 K_2$  is the minimal normal subloop of  $Q$  containing the normal subloops  $K_1, K_2$ . Hence the ideal  $I_1 + I_2$  induces the normal subloop  $K_1 K_2$  of loop  $Q$ .

The loop  $Q$  is simple. Then  $K_1 = K_2 = 1$ . Hence the sum  $I_1 + I_2$  of any proper ideals  $I_1, I_2$  of  $F\{Q\}$  induces in loop  $Q$  the unitary subloop 1, i.e. induces the identical mapping in  $Q$ .

It is easy to see that the sum  $S$  of all proper ideals of  $F\{Q\}$  induces in loop  $Q$  the identical mapping in  $Q$ . Hence the algebra  $F(Q) = F\{Q\}/S$  is non-trivial, simple and  $Q$  is a subloop of loop  $U(F(Q))$ . The elements  $g \in Q$  are invertible, then the elements  $1 - g$  are quasiregular in algebra  $F(Q)$ . Hence the Smiley radical  $\mathcal{S}(F(Q))$  of algebra  $F(Q)$  is non-trivial,  $\mathcal{S}(F(Q)) \neq (0)$ . Remind that the Smiley radical  $\mathcal{S}(A)$  of an alternative algebra  $A$  consists from all quasiregular elements of  $A$ .

Any simple alternative algebra is Cayley-Dickson algebra  $\mathbf{C}(\mu, \beta, \gamma) = \mathbf{C}$  over their centre  $Z(\mathbf{C})$ . As  $1 \in Z(\mathbf{C})$  then  $Z(\mathbf{C}) \neq 0$ . In such case the centre  $Z(\mathbf{C})$  is a field. Then  $F$  is a field.

Any Cayley-Dickson algebra  $\mathbf{C}$  is primitive. The Kleinfeld radical  $\mathcal{K}(A)$  of alternative algebra  $A$  is the intersection of all ideals  $I$  of  $A$  such that the quotient-algebra  $A/I$  is a primitive algebra. As the algebra  $F(Q)$  is simple then  $(0)$  is the unique maximal ideal of  $F(Q)$ . Hence  $\mathcal{K}(F(Q)) = (0)$ .

The Kleinfeld radical  $\mathcal{K}(A)$  coincides with Smiley radical  $\mathcal{S}(A)$  in any alternative algebra  $A$ ,  $\mathcal{K}(A) = \mathcal{S}(A)$ . Then from  $\mathcal{K}(F(Q)) = (0)$  it follows that  $\mathcal{S}(F(Q)) = (0)$ . But before we proved that  $\mathcal{S}(F(Q)) \neq (0)$ . We get a contradiction. Hence our supposition that the loop  $Q$  is imbedded into a loop  $\mathcal{U}(A)$  for a certain alternative algebra is false. This completes the proof of Theorem.

**Remark.** The question on embedding of Moufang loops into alternative algebras is examined and in Theorem 1 from [7]. But with regret the statement of these theorem is not correct, is not in line with the proof. Maybe the translation into English is not correct. Without getting into details, this theorem is a consequence of the following statement that will be published in the following papers of the author: the Moufang loops considered in Theorem of this paper they and only they are not imbedded into the loop of type  $\mathcal{U}(A)$  for a suitable unital alternative algebra.

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